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EMBEDDINGS OF LOCALLY CONNECTED COMPACTA

BY

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ABSTRACT. Let X be a k-dimensional compactum and $f\colon X \to M^n$ a map into a piecewise linear n-manifold, $n \geqslant k + 3$. The main result of this paper asserts that if X is locally $(2k - n)$-connected and f is $(2k - n + 1)$-connected, then f is homotopic to a CE equivalence. In particular, every k-dimensional, r-connected, locally r-connected compactum is CE equivalent to a compact subset of $\mathbf{R}^{2k-r}$ as long as $r \leqslant k - 3$.

Introduction. Let X be a k-dimensional compactum. In this paper we study the problem of finding an embedding of X into Euclidean n-space $\mathbf{R}^n$. Specifically, we wish to investigate conditions under which we can embed X in $\mathbf{R}^n$ when n is less than the classical dimension $2k + 1$.

In case X is a manifold, it has long been known that such improvements are possible. Whitney **[Wh]** showed that every smooth k-dimensional manifold embeds in $\mathbf{R}^{2k}$. His techniques were later generalized to codimension three embedding theorems for piecewise linear (PL) manifolds by Irwin **[Ir]** and Hudson **[Hd]** who showed that if the manifold is r-connected, then it is possible to PL embed it in $\mathbf{R}^{2k-r}$.

The situation for polyhedra is different. In order to accomplish the embedding into a lower-dimensional space it is necessary to identify certain contractible sets to points and this changes the homeomorphism type of the polyhedron. Thus the map produced is not a topological embedding, but rather a simple homotopy equivalence. So the appropriate theory for polyhedra is a theory of embedding up to simple homotopy type and this theory was worked out by Stallings **[St]**. It follows from his main theorem that every k-dimensional, r-connected polyhedron is simple homotopy equivalent to a subpolyhedron of $\mathbf{R}^{2k-r}$.

The purpose of this paper is to prove a theorem like that of Stallings, but for more general compacta rather than polyhedra. In moving from polyhedra to compacta it is necessary to add an additional hypothesis: We must assume that the polyhedra are locally r-connected as well as globally r-connected. In addition, we must use an appropriate generalization of the (strictly polyhedral) concept of simple homotopy equivalence. We substitute CE equivalence. This is the natural thing to do because

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every CE map of polyhedra induces a simple homotopy equivalence [**Ch**] and a simple homotopy equivalence is made up of a sequence of maps, each of which shrinks out cell-like sets of a particular kind.

Before stating our main theorem we give a definition of CE equivalence.

DEFINITION. A compact space A is said to be a *cell-like set* if, for some embedding of A into an absolute neighborhood retract (ANR), A has the property that A is contractible in every neighborhood of itself. A map $f\colon X \to Y$ of compacta is said to be a *CE map* if f is onto and $f^{-1}(y)$ is a cell-like subset of X for every $y \in Y$. We say that f is a *CE equivalence* if there exists a compactum Z and CE maps $g\colon Z \to X$ and $h\colon Z \to Y$ such that $h = fg$.

THEOREM 1. *Suppose X is a locally r-connected, k-dimensional compactum and M^n is a PL n-manifold, $k \leqslant n - 3$, $r \geqslant 2k - n$. If $f\colon X \to M$ is an $(r + 1)$-connected map, then f is homotopic to a CE equivalence; i.e., there exist k-dimensional compacta Y and Z such that $Y \subset M^n$ and CE maps $Z \to X$ and $Z \to Y$ such that*

$$\begin{array}{ccc} X & \xrightarrow{f} & M \\ \uparrow & & \cup \\ Z & \to & Y \end{array}$$

commutes up to homotopy. Furthermore, if $r \leqslant k - 2$ then Y is locally r-connected.

COROLLARY. *If X is a k-dimensional, r-connected, locally r-connected compactum, $r \leqslant k - 3$, then X is CE equivalent to a compact, k-dimensional, locally r-connected subset of $\mathbf{R}^{2k-r}$.*

We note that Theorem 1 has precisely the same connectivity conditions as are found in Stallings' theorem, except that we assume local connectivity as well as global connectivity. The local connectivity is automatic in the case of polyhedra and the example in [**D-H**] shows that it is a necessary hypothesis in the case of compacta.

There have been other generalizations of [**St**] to compacta. One such generalization involves the use of shape equivalence instead of CE equivalence. The main results of that type are contained in [**H-I**]. It is interesting to note that the hypotheses used in [**H-I**] are equivalent (by [**Fe1**]) to the assumption that X has the shape of an LC^r compactum for precisely the same value of r as is used here.

In addition, Husch [**Hs**] has proved a metastable range theorem similar to ours but using a condition which is stronger than the LC^r hypothesis. He proves that if X can be expanded as the inverse limit of an inverse sequence of k-dimensional polyhedra with UV^r bonding maps, $r \geqslant 2k - n$, $k \leqslant n - 3$, and $k < (2/3)(n - 1)$, then any $(r + 1)$-connected map $f\colon X \to M^n$ is homotopic to a CE equivalence. Ferry [**Fe1**] has shown that if X can be written as an inverse limit of an inverse sequence with UV^r bonding maps, then X is LC^r (while the converse is not true). Thus Husch's theorem follows from Theorem 1. But the proof of Theorem 1 given in this paper is based on Husch's construction.

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1. Definitions. A space X is said to be *r-connected* if every map of the m-sphere S^m, $m \leqslant r$, into X extends to a map of the $(m+1)$-ball B^{m+1} into X. We say that X is *locally r-connected* (abbreviated LC^r) if for every $x \in X$ and for every neighborhood U of x there exists a neighborhood V of x such that any map of S^m into V extends to a map of B^{m+1} into U, $m \leqslant r$. A map $f\colon X \to Y$ is said to be an *r-connected map* if $\pi_i(M(f), X) = 0$ for $i \leqslant r$. Here $M(f)$ denotes the mapping cylinder of f.

Let ε be a positive number. A homotopy $f\colon X \times [0,1] \to Y$ is said to be an ε-homotopy if $\operatorname{diam}(f(\{x\} \times [0,1])) < \varepsilon$ for every $x \in X$. A collapse of one polyhedron to another induces a strong deformation retraction in a natural way. The collapse is said to be an *ε-collapse* if the induced homotopy is an ε-homotopy. A regular neighborhood N of a polyhedron K is said to be an *ε-regular neighborhood* if N collapses to K via an ε-collapse.

All polyhedra considered in this paper are compact. If L is a polyhedron equipped with a triangulation, then $L^{(r)}$ denotes the r-skeleton.

Inverse limits are useful in this paper. If $\{X_i\}$ is a sequence of spaces and $f_i\colon X_{i+1} \to X_i$, we consider $\varprojlim \{X_i, f_i\}$ to be the subset of the product space $\prod X_i$ consisting of all sequences (x_i) such that $f_i(x_{i+1}) = x_i$.

It is also convenient to use shape theory occasionally to prove that certain sets are cell-like. We use only the most elementary facts which can be found in any book on the subject (see **[D-S]**, for example).

Finally, we mention taming theory. We follow **[Ed** and **Št]** and say that a k-dimensional compactum X in a PL manifold is *tame* if for every $\varepsilon > 0$ there exists a neighborhood N of X such that N is an ε-regular neighborhood of a k-dimensional polyhedron.

2. Neighborhoods. Let X denote a k-dimensional, compact, LC^r metric space which is a subset of the interior of a PL n-manifold M. In this section we will build neighborhoods of X in M which reflect the local connectivity of X. Specifically, this means that the neighborhoods should contain small homotopies pushing polyhedra of appropriate dimension into X.

DEFINITION. Let ε be a small positive number. We will call a neighborhood N of X an *(ε, r)-deformation neighborhood* if for every $(r+1)$-dimensional polyhedron L and every map $f\colon L \to N$ there exists an ε-homotopy $f_t\colon L \to N$ such that $f_0 = f$ and $f_1(L) \subset X$. We say that N is a *strong (ε, r)-deformation neighborhood* if N has the additional property that if L_0 is a subpolyhedron of L such that $f(L_0) \subset X$, then we can require that $f_t|L_0 = f|L_0$ for every t.

In this section we will prove that, under suitable hypotheses, LC^r compacta have arbitrarily small strong (ε, r)-deformation neighborhoods. The converse is also true: if X has strong (ε, r)-deformation neighborhoods for arbitrarily small ε, then X is

LC^r. This is because a map of S^m into a small subset of X extends to a map of B^{m+1} into a small subset of M. If $m \leqslant r$, a strong (ε, r)-deformation neighborhood can then be used to push the image of B^{m+1} into X.

It seems simplest to begin by proving the existence of (weak) (ε, r)-deformation neighborhoods. The proof of the following lemma was communicated to the author by M. Bestvina.

LEMMA 2.1. *Suppose X is a k-dimensional compactum which is tamely embedded in the PL n-manifold M. If X is LC^r, $r \leqslant k-2$, $k+r+3 \leqslant n$, then for every $\varepsilon > 0$ there is an (ε, r)-deformation neighborhood N of X in M such that N is an ε-regular neighborhood of a k-dimensional polyhedron K.*

PROOF. The proof is by induction on r and begins with the case $r = -1$ (in which case there is something to prove even though there is no restriction on X).

Choose $\delta > 0$ such that for any point $x \in N_\delta(X)$ there exists a path of diameter $< \varepsilon/5$ from x to a point of X. Let N_1 be a neighborhood of X such that N_1 is an $(\varepsilon/5)$-regular neighborhood of a k-dimensional polyhedron K_1 and $N_1 \subset N_\delta(X)$. Take a triangulation of K_1 having mesh less than $\varepsilon/5$.

For each vertex v of K_1, choose an $(\varepsilon/5)$-arc B_v from v to a point of X. We may assume that $B_v \cap K_1 = \{v\}$. (If the other end of B_v lies in $K_1 \cap X$, just move K_1 slightly.) Keep the endpoints of B_v fixed and push the rest of B_v out of N_1 using the natural $(\varepsilon/5)$-product structure on $N_1 - K_1 \cong \partial N_1 \times [0,1)$. The new arc, C_v, has diameter $< 3\varepsilon/5$. Now add a small $(\varepsilon/5)$-regular neighborhood of each arc C_v to N_1. We take N to be N_1 plus the neighborhoods of the arcs and take K to be K_1 plus the arcs C_v plus the shadows of $C_v \cap \partial N_1$ under the collapse of N_1 to K_1.

Pick a point $p \in N$. We must show that there is an ε-homotopy of p in N which pushes p into X. First homotope p into K with an $(\varepsilon/5)$-homotopy. In case this pushes p into K_1, continue to push p along K_1 to the nearest vertex. The other case is that the first homotopy pushes p into one of the arcs C_v. In either case we get p into one of the C_v with a $(2\varepsilon/5)$-homotopy. The homotopy is completed by moving along C_v into X.

Now suppose that $r \geqslant 0$ and that X has $(\varepsilon, r-1)$-deformation neighborhoods for arbitrarily small ε.

Let $\varepsilon > 0$ be given. Choose $\delta_1 > 0$ such that any δ_1-map of S^{r+1} into the δ_1-neighborhood of X extends to an $(\varepsilon/8)$-map of B^{r+2} into M. (The existence of δ_1 follows from the fact that M is LC^{r+1}.) Next we use the fact that X is LC^r to choose $\delta > 0$ such that $\delta < \delta_1/2$ and if L is any $(r+1)$-complex and $f\colon L^{(r)} \to X$ such that $\operatorname{diam}[f(\partial\Delta)] < \delta$ for every $(r+1)$-simplex Δ in L, then f extends to a map $f'\colon L \to X$ such that $\operatorname{diam}[f'(\Delta)] < \delta_1/2$ for every simplex Δ in L.

Now apply the inductive hypothesis to get a $(\delta/3, r-1)$-deformation neighborhood N_1 of X such that N_1 collapses to a k-dimensional polyhedron K_1 via an $(\varepsilon/8)$-collapse. Triangulate K_1 with a triangulation of mesh less than $\delta/3$ and let $K_1^{(r+1)}$ denote the $(r+1)$-skeleton of K_1.

We claim that there exists an $(\varepsilon/8)$-homotopy of $K_1^{(r+1)}$ into X which keeps $K_1^{(r)}$ in N_1. By the choice of N_1, there exists a $(\delta/3)$-homotopy $h_t\colon K_1^{(r)} \to N_1$ such that $h_0 =$ inclusion and $h_1(K_1^{(r)}) \subset X$. Extend h_0 via the inclusion to all of $K_1^{(r+1)}$. For each $(r+1)$-simplex $\Delta \subset K_1^{(r+1)}$, $h_1(\partial\Delta) \subset X$ and $\operatorname{diam}[h_1(\partial\Delta)] < \delta$. So the choice of δ allows us to extend h_1 to $K_1^{(r+1)}$ in such a way that $h_1(K_1^{(r+1)}) \subseteq X$ and $\operatorname{diam}[h_1(\Delta)] < \delta_1/2$ for each $\Delta \subseteq K_1^{(r+1)}$. Then for each $(r+1)$-simplex $\Delta \subseteq K_1^{(r+1)}$, $h_0(\Delta) \cup h_1(\Delta) \cup \{h_t(\partial\Delta) | 0 \leqslant t \leqslant 1\}$ has diameter less than δ_1. The choice of δ_1 now allows us to extend h_t to the $(\varepsilon/8)$-homotopy whose existence was claimed above.

We now have a map h: $K_1^{(r+1)} \times [0,1] \to M$ such that $h(x,0) = x$ for all $x \in K_1^{(r+1)}$, $h(K_1^{(r+1)} \times \{1\}) \subseteq X$, $h(K_1^{(r)} \times [0,1]) \subseteq N_1$, and $\operatorname{diam}[h(\Delta \times [0,1])] < \varepsilon/8$ for every $(r+1)$-simplex $\Delta \subset K_1^{(r+1)}$. For some small positive number γ, make $h|K_1^{(r+1)} \times [0, 1-\gamma]$ PL and put it in general position with respect to K_1. The condition $k + r + 3 \leqslant n$ then implies that $h(K_1^{(r+1)} \times [0, 1-\gamma])$ intersects K_1 only along $h(K_1^{(r+1)} \times \{0\}) = K_1^{(r+1)}$. We may also assume that γ and the PL approximation were chosen carefully enough so that

$$h\left(K_1^{(r+1)} \times \{0, 1-\gamma\}\right) \cup h\left(K_1^{(r)} \times [0, 1-\gamma]\right) \subset N_1.$$

Push the rest of $h(K_1^{(r+1)} \times [0, 1-\gamma])$ out of N_1, keeping $h(K_1^{(r+1)} \times \{0, 1-\gamma\}) \cup h(K_1^{(r)} \times [0, 1-\gamma])$ fixed. Then $h(K_1^{(r+1)} \times [0, 1-\gamma])$ consists of the part lying in N_1 together with some $(r+2)$-cells poking out of N_1. Each of these $(r+2)$-cells has diameter less than $3\varepsilon/8$. We take N to be N_1 plus $(\varepsilon/8)$-regular neighborhoods of the $(r+2)$-cells and take K to be K_1 plus the $(r+2)$-cells plus shadows of the $(r+2)$-cells in the product structure of $N_1 - K_1$.

We must now show that N satisfies the conclusion of the lemma. Let L be an $(r+1)$-dimensional polyhedron and f: $L \to N$. We construct the homotopy of $f(L)$ into X in several stages. First, homotope $f(L)$ out of the $(r+2)$-handles and into N_1. Next, push $f(L)$ down the product structure of $N_1 - K_1$ into K_1. Then push along K_1 into $K_1^{(r+1)}$. From there on, use the homotopy h_t. The second and third of those homotopies are $(\varepsilon/8)$-homotopies and the first and last are $(3\varepsilon/8)$-homotopies, so the entire homotopy is an ε-homotopy. □

LEMMA 2.2. *Suppose X is a compact LC^0 subset of the PL manifold M. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every $(\delta, 0)$-deformation neighborhood of X is a strong $(\varepsilon, 0)$-deformation neighborhood of X.*

PROOF. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $3\delta \leqslant \varepsilon/6$ and such that if x, $y \in X$ and $d(x, y) < 3\delta$, then x and y can be joined by an arc in X of diameter less than $\varepsilon/6$.

Now suppose that N is a $(\delta, 0)$-deformation neighborhood of X in M. Let L be a 1-dimensional polyhedron, L_0 a subpolyhedron of L and f: $L \to N$ a map such that $f(L_0) \subseteq X$. Choose a triangulation of L of such small mesh that $\operatorname{diam}[f(\Delta)] < \delta$ for every simplex Δ in L. For each vertex v of $L - L_0$, there is a δ-homotopy of $f(v)$ into X. Extend those homotopies via the identity to $f(L_0)$ and then to a δ-homotopy of all of $f(L)$ in N.

Now if Δ is a 1-simplex in L, we have δ-homotoped $f|\Delta$ to a map f': $\Delta \to N$ such that $f'(\partial\Delta) \subseteq X$ and $\operatorname{diam}[f'(\Delta)] < 3\delta \leqslant \varepsilon/6$. Thus the proof of the lemma will be complete if we can show that f' is $(5\varepsilon/6)$-homotopic, rel $\partial\Delta$, to a map of Δ into X. To do so we use a base point trick just like that in [**L-V**, Lemma 1].

By the choice of δ, we can extend $f'|\partial\Delta$ to a map f'': $\Delta \to X$ such that $\operatorname{diam}[f''(\Delta)] < \varepsilon/6$. Let Ω be the loop made up of $f'(\Delta)$ together with $f''(\Delta)$ with reverse orientation. We want to show that Ω is homotopic, rel base point, to a loop Ω' in X. If so, then $f'(\Delta)$ is homotopic (rel base point) to the path $f''(\Delta)$ followed by Ω' which lies entirely in X and so we are finished.

Take a triangulation of N of mesh less than δ and let K denote the 1-skeleton of the triangulation. By hypothesis, there exists a δ-homotopy h_t: $K \to N$ such that $h_0 =$ inclusion and $h_1(K) \subset X$. By moving f'' slightly (if necessary) we may assume that there is a point $y \in K$ such that $h_1(y) \in f''(\Delta)$. This may increase the size of $f''(\Delta)$, but only by $\varepsilon/6$. We will first use y as a base point. Let a be the path from y to $h_1(y)$ traced out by the homotopy $h_t|\{y\}$. Notice that there is a base-point-fixing 2δ-homotopy of the loop $a\Omega a^{-1}$ to a loop of the form $a\Omega' a^{-1}$ where Ω' is a loop in X. (This homotopy consists of a push into the 1-skeleton K, followed by the homotopy h_t.) But now using $h_1(y)$ as a base point, we see that $\Omega \simeq a^{-1}a\Omega a^{-1}a \simeq a^{-1}a\Omega' a^{-1}a \simeq \Omega'$ where "$\simeq$" denotes "is homotopic to" and all the homotopies are rel the base point $h_1(y)$. To complete the proof we need only compute the size of the homotopies involved. The entire image of the homotopy has

$$\begin{aligned}\text{diameter} &\leqslant \operatorname{diam}[\Omega] + \operatorname{diam}[a] + 4\delta + \operatorname{diam}[a]\\ &\leqslant \left(\operatorname{diam}\left[f'(\Delta)\right] + \operatorname{diam}\left[f''(\Delta)\right] + \varepsilon/6\right) + \delta + 4\delta + \delta\\ &\leqslant 3\delta + \varepsilon/6 + \varepsilon/6 + 6\delta < 5\varepsilon/6. \qquad \square\end{aligned}$$

The following is the main result of this section.

PROPOSITION 2.3. *Suppose X is a k-dimensional compactum which is tamely embedded in the PL n-manifold M. If X is LC^r, $r \leqslant k - 2$, and $k + r + 3 \leqslant n$, then for every $\varepsilon > 0$ there is a strong (ε, r)-deformation neighborhood N of X in M such that N is an ε-regular neighborhood of a k-dimensional polyhedron K.*

REMARK. The hypothesis $r + k + 3 \leqslant n$ could be weakened considerably. But doing so would require extra work and we do not have any need for a stronger result in this paper.

PROOF. The proof is by induction on r. The case $r = -1$ has already been done because there is no difference between an $(\varepsilon, -1)$-deformation neighborhood and a strong $(\varepsilon, -1)$-deformation neighborhood. The case $r = 0$ is covered by Lemmas 2.1 and 2.2. So we may assume inductively that $r \geqslant 1$ and that for every $\varepsilon > 0$ there is a $\delta > 0$ such that every $(\delta, r - 1)$-deformation neighborhood of X is a strong $(\varepsilon, r - 1)$-deformation neighborhood of X.

Let $\varepsilon > 0$ be given. Use the fact that X is LC^r to choose $\delta_1 > 0$ such that any map g: $S^r \to X$ with $\operatorname{diam}[g(S^r)] < 3\delta_1$ extends to a map G: $B^{r+1} \to X$ such that $\operatorname{diam}[G(B^{r+1})] < \varepsilon/5$. By the inductive hypothesis there exists $\delta > 0$ such that any $(\delta, r - 1)$-deformation neighborhood of X is a strong $(\delta_1, r - 1)$-deformation neighborhood. We may also assume that $\delta \leqslant \delta_1$ and $3\delta_1 \leqslant \varepsilon/5$.

By Lemma 2.1 there exists a neighborhood N of X such that N is a (δ, r)-deformation neighborhood and such that N is a δ-regular neighborhood of a k-dimensional polyhedron K. We will show that such an N is a strong (ε, r)-deformation neighborhood.

Let L be an $(r+1)$-dimensional polyhedron, L_0 a subpolyhedron of L and f: $L \to N$ a map such that $f(L_0) \subset X$. We must produce an ε-homotopy f_t: $L \to N$ such that $f_0 = f$, $f_1(L) \subset X$, and $f_t|L_0 = f|L_0$ for every t. We may assume that L_0 is r-dimensional, because otherwise we could restrict our attention to $\mathrm{Clos}(L - L_0)$, construct such a homotopy of $\mathrm{Clos}(L - L_0)$ which keeps $L_0 \cap \mathrm{Clos}(L - L_0)$ fixed and then extend it via f to all of $L = \mathrm{Clos}(L - L_0) \cup L_0$.

Pick a triangulation for L such that if Δ is a simplex in L, then $\mathrm{diam}[f(\Delta)] < \delta$ and L_0 is a subcomplex of this triangulation. By the choice of N, there is a δ_1-homotopy h_t: $L \to N$ such that $h_0 = f$, $h_t|L_0 = f|L_0$ for each t, and $h_1(L^{(r)}) \subset X$. For each $(r+1)$-simplex Δ of L, we have that $h_1(\partial\Delta) \subset X$ and $\mathrm{diam}[h_1(\Delta)] < 3\delta_1$. Just as in the proof of Lemma 2.2, we will be finished if we can show that there is a $(4\varepsilon/5)$-homotopy of $h_1(\Delta)$ into X which keeps $h_1(\partial\Delta)$ fixed.

So consider one such Δ. By the choice of δ_1, there is a map h': $\Delta \to X$ which extends $h_1|\partial\Delta$ and such that $\mathrm{diam}[h'(\Delta)] < \varepsilon/5$. Consider the singular $(r+1)$-sphere $S = h_1(\Delta) \cup h'(\Delta)$. There is a δ-homotopy of S to a singular $(r+1)$-sphere $S' \subset X$. Pick a base point $x \in S \cap X$. The path p followed by x during the homotopy from S to S' is a δ-path beginning and ending on X. By induction, there exists a δ_1-homotopy of p into X which keeps the ends of p fixed. Let p' be the new path in X. Notice that $\mathrm{diam}[p'] \leqslant \delta + 2\delta_1$ and that S is homotopic (rel base point) to the singular $(r+1)$-sphere $S'' \subset X$ which is S' acted on by the path p'. Therefore $h_1|\Delta$ is homotopic, rel $\partial\Delta$, to a map of Δ onto $S'' \cup h'(\Delta)$.

The size of that homotopy is no greater than $\mathrm{diam}[S] + 2\delta + 2\delta_1$. But $\mathrm{diam}[S] \leqslant 3\delta_1 + \varepsilon/5$, so the size of the homotopy is no more than $3\delta_1 + \varepsilon/5 + 2\delta + 2\delta_1 < 4\varepsilon/5$. □

Proposition 2.3 seems like the simplest, most natural way to state the properties of the neighborhoods we have constructed. But in the proof of Theorem 1 we will actually need the following, more technical, statement.

Proposition 2.4. *Let X be an LC^r compactum in the PL n-manifold M and let ε be a positive number. Suppose N_1 is a strong (ε, r)-deformation neighborhood of X, ρ: $N_1 \to K_1$ is an ε-retraction onto a k-dimensional spine K_1, $N_0 \subseteq N_1$ is a second (ε, r)-deformation neighborhood of X, L is an $(r+1)$-dimensional polyhedron and L_0 is a subpolyhedron of L. Then if f_0: $L_0 \to N_0$ is a map such that ρf_0 extends to a map f: $L \to K_1$, then there is an extension f': $L \to N_0$ of f_0 such that f and $\rho f'$ are 5ε-homotopic, rel L_0, in K_1.*

Proof. We will show that there is a 3ε-homotopy g_t: $L \to N_1$ such that $g_0 = f$, $g_1(L) \subset N_0$, $g_1|L_0 = f_0$ and $\rho(g_t(x)) = f(x)$ for every $x \in L_0$ and for every t. Once such a homotopy has been constructed, we simply take $f' = g_1$ and notice that the homotopy ρg_t is a 5ε-homotopy from $\rho g_0 = f$ to $\rho g_1 = \rho f'$.

To construct g_t we proceed as follows. Pick a barycentric subdivision of L having very small mesh. Let A denote the simplicial neighborhood of L_0 in this triangulation, $C = \mathrm{Clos}(L - A)$ and $B = A \cap C$. Since N_0 is an (ε, r)-deformation neighborhood, there exists an ε-homotopy of $f(L_0)$ in N_0 which pushes $f(L_0)$ into X. By spreading that homotopy out over the neighborhood A, we construct a map F: $L \times [0,1] \to N_1$ such that $F(x,0) = f(x)$ for all $x \in L$, $F(x,1) = f_0(x)$ for all $x \in L_0$, $F(x,1) \in N_0$ for all $x \in A$, $F(x,1) \in X$ for all $x \in B$, $\rho(F(x,t)) = f(x)$ for all $x \in L_0$ and all t, and $\operatorname{diam}[F(\{x\} \times [0,1])] < 2\varepsilon$ for all t. Now there exists an ε-homotopy which pushes $F(C \times \{1\})$ into X keeping $F(B \times \{1\})$ fixed because of the fact that N_1 is a strong (ε, r)-deformation neighborhood. The homotopy g_t is the homotopy obtained by doing the two homotopies mentioned above in succession. □

3. Inverse limits and CE limit maps. In this section we give conditions under which a sequence of CE maps into neighborhoods of X converges to a CE map onto X. We first show that if X is defined as the intersection of a sequence of neighborhoods with sufficiently small collapses onto their spines, then there is a natural CE map from the inverse limit of the spines onto X. Then we construct a CE map from an inverse limit onto X as the limit of a sequence of level maps into that inverse sequence.

The following notation is assumed throughout this section: X is a compactum in the interior of the PL n-manifold M, $\{N_i\}$ is a sequence of neighborhoods of X, $N_{i+1} \subset \operatorname{Int} N_i$ for each i, $X = \bigcap N_i$, N_i collapses to a compact polyhedron X_i via an ε_i-collapse ξ_i, and $\{\varepsilon_i\}$ is a sequence of positive numbers which converges to 0. Let ρ_i: $N_i \to X_i$ denote the ε_i-retraction induced by ξ_i and let f_i: $X_{i+1} \to X_i$ be the restriction of ρ_i. We will refer to the sequence $\{N_i, X_i\}$ as a *defining sequence* for X. Notice that the defining sequence $\{N_i, X_i\}$ implicitly defines the maps f_i and the numbers ε_i.

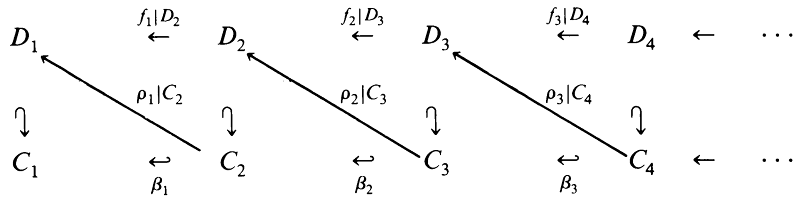
LEMMA 3.1. *Suppose X is a compact subset of the PL n-manifold M. If $\{N_i, X_i\}$ is a defining sequence for X such that $\sum \varepsilon_i < \infty$, then there is a natural CE map G: $\varprojlim \{X_i, f_i\} \to X$.*

REMARK. It is possible to make G a homeomorphism by putting further restrictions on the defining sequence. To do so, the ε_i would have to be chosen inductively and would depend on the preceding retractions. A proof of this stronger lemma would follow the lines of the proofs of Theorems 1 and 2 in **[Br]**.

PROOF OF LEMMA 3.1. Let $(x_i) \in \varprojlim \{X_i\}$. The facts that $d(x_i, x_{i+1}) < \varepsilon_i$ and $\sum \varepsilon_i < \infty$ imply that the sequence $\{x_i\}$ forms a Cauchy sequence when considered as a sequence of points in M. Therefore there is a unique point y such that $x_i \to y$ as $i \to \infty$. Notice that y must be a point in X, so we can define $G((x_i)) = y$. It is clear that G is continuous, so the proof will be complete if we show that $G^{-1}(x)$ is a nonempty cell-like set for each $x \in X$.

Fix $x \in X$. Define $C_i = \{y \in N_i | d(y,x) < \sum_{k \geq i} \varepsilon_k\}$ and $D_i = C_i \cap X_i$. It is clear that $\bigcap C_i = \{x\}$; i.e., if β_i: $C_{i+1} \to C_i$ denotes the inclusion map, then $\{x\} = \varprojlim \{C_i, \beta_i\}$. Pick $y \in D_{i+1}$; then $y \in X_{i+1}$ and $d(x,y) < \sum_{k \geq i+1} \varepsilon_k$, so $d(x, f_i(y)) < \sum_{k \geq i} \varepsilon_k$. Thus $f_i(D_{i+1}) \subset D_i$. We finish the proof by showing the following: $G^{-1}(x) = \varprojlim \{D_i, f_i | D_{i+1}\}$ and $\operatorname{Sh}(\varprojlim \{D_i, f_i | D_{i+1}\}) = \operatorname{Sh}(\varprojlim \{C_i, \beta_i\})$. This will complete the proof because then $G^{-1}(x)$ is nonempty (it is the inverse limit of an inverse sequence of nonempty compacta) and it is cell-like because it has the shape of a point.

Consider the diagram:



There is an ε_i-homotopy from β_i to $\rho_i|C_{i+1}$. That homotopy stays in the ε_i-neighborhood of C_{i+1}, and thus stays in C_i. Therefore each of the lower triangles in the diagram commutes up to homotopy while each of the upper triangles commutes exactly. Thus we have a shape equivalence from $\varprojlim\{D_i, f_i|D_{i+1}\}$ to $\varprojlim\{C_i, \beta_i\}$.

We now show that $G^{-1}(x) = \varprojlim\{D_i, f_i|D_{i+1}\}$. Pick $(x_i) \in G^{-1}(x)$. Then $d(x_i, x_{i+1}) < \varepsilon_i$ and $x_i \to x$. Thus $d(x_i, x) \leqslant \sum_{k \geqslant i} \varepsilon_k$ and $x_i \in D_i$. On the other hand, if $(x_i) \in \varprojlim\{D_i\}$, then $d(x_i, x) \leqslant \sum_{k \geqslant i} \varepsilon_k$. So $x_i \to x$ as $i \to \infty$ and $(x_i) \in G^{-1}(x)$. □

Remark. Since every PL manifold has a metric in which closed balls are subpolyhedra, we may assume that the D_i in the proof above are subpolyhedra.

The following proposition is similar to Proposition 15 of **[Hs]**. The main difference is that we do not assume that the bonding maps in the inverse sequences are onto, but use Lemma 3.1 instead.

Proposition 3.2. *Let X be a compact subset of the PL n-manifold M and let $\{N_i, X_i\}$ be a defining sequence for X such that $\sum \varepsilon_i < \infty$. Suppose $\{Z_i, g_i\}$ is an inverse sequence of compact polyhedra and that for each i there is a PL CE map λ_i: $Z_i \to X_i$. If $f_i\lambda_{i+1}$ is α_i-homotopic to $\lambda_i g_i$ for each i and $\sum \alpha_i < \infty$, then Λ: $\varprojlim\{Z_i, g_i\} \to X$ defined by $\Lambda((z_i)) = \lim_{i\to\infty} \lambda_i(z_i)$ is a CE map.*

Proof. Let $Z = \varprojlim\{Z_i, g_i\}$. Define Λ_j: $Z \to N_1$ by $\Lambda_j((z_i)) = \lambda_j(z_j)$. Then

$$\begin{aligned} d\big(\Lambda_{j+1}((z_i)), \Lambda_j((z_i))\big) &= d\big(\lambda_{j+1}(z_{j+1}), \lambda_j(z_j)\big) = d\big(\lambda_{j+1}(z_{j+1}), \lambda_j g_j(z_{j+1})\big) \\ &\leqslant d\big(\lambda_{j+1}(z_{j+1}), f_j\lambda_{j+1}(z_{j+1})\big) + d\big(f_j\lambda_{j+1}(z_{j+1}), \lambda_j g_j(z_{j+1})\big) \\ &\leqslant \varepsilon_j + \alpha_j. \end{aligned}$$

Thus $\{\Lambda_j\}$ forms a Cauchy sequence of maps and therefore converges to a continuous map Λ: $Z \to X$.

Fix $x \in X$. To complete the proof we must show that $\Lambda^{-1}(x)$ is a nonempty cell-like subset of Z. Define $D_i = \{y \in X_i | d(x, y) \leqslant \sum_{k \geqslant i}(\varepsilon_k + \alpha_k)\}$ and $E_i = \lambda_i^{-1}(D_i)$. Notice that $f_i(D_{i+1}) \subset D_i$. It is also true that $g_i(E_{i+1}) \subset E_i$. To see this, choose $w \in E_{i+1}$. We must show that $g_i(w) \in E_i = \lambda_i^{-1}(D_i)$ which is equivalent to showing that $\lambda_i g_i(w) \in D_i$.

$$\begin{aligned} d(\lambda_i g_i(w), x) &\leqslant d(\lambda_i g_i(w), f_i\lambda_{i+1}(w)) + d(f_i\lambda_{i+1}(w), x) \\ &\leqslant \alpha_i + [\varepsilon_i + d(\lambda_{i+1}(w), x)] \\ &\leqslant \alpha_i + \varepsilon_i + \sum_{k \geqslant i+1} (\varepsilon_k + \alpha_k) = \sum_{k \geqslant i} (\alpha_k + \varepsilon_k). \end{aligned}$$

Thus $\lambda_i g_i(w) \in E_i$.

Let G be the map defined in the proof of Lemma 3.1. We make the following three claims.

(i) $\Lambda^{-1}(x) = \varprojlim\{E_i, g_i|E_{i+1}\}$.

(ii) $\mathrm{Sh}(\varprojlim\{E_i, g_i|E_{i+1}\}) = \mathrm{Sh}(\varprojlim\{D_i, f_i|D_{i+1}\})$.

(iii) $\varprojlim\{D_i, f_i|D_{i+1}\} = G^{-1}(x)$.

Once claims (i)–(iii) are verified, the proof is complete.

Let $w \in \Lambda^{-1}(x)$. Then $w = (w_i)$ and $\lambda_i(w_i) \to x$ as $i \to \infty$. But

$$d(\lambda_{i+1}(w_{i+1}), \lambda_i(w_i)) \leqslant \alpha_i + \varepsilon_i$$

(by the calculation at the beginning of this proof) and so $w_i \in E_i$. On the other hand, if $(w_i) \in \varprojlim\{E_i\}$, then $d(\lambda_i(w_i), x) \leqslant \varepsilon_i + \alpha_i$, so $\lambda_i(w_i) \to x$ and $(w_i) \in \Lambda^{-1}(x)$. Thus claim (i) is verified.

The following diagram is homotopy commutative for each i.

$$\begin{array}{ccc} D_i & \xleftarrow{f_i|D_{i+1}} & D_{i+1} \\ \uparrow \lambda_i|E_i & & \uparrow \lambda_{i+1}|E_{i+1} \\ E_i & \xleftarrow[g_i|E_{i+1}]{} & E_{i+1} \end{array}$$

Furthermore, $\lambda_i|E_i = \lambda_i|\lambda_i^{-1}(D_i)$ is a homotopy equivalence [**La**, Theorem 1.2]. Therefore $\{\lambda_i|E_i\}$ induces a shape equivalence from $\varprojlim\{E_i, g_i|E_{i+1}\}$ to $\varprojlim\{D_i, f_i|D_{i+1}\}$ and claim (ii) is verified.

The proof of claim (iii) is just like the proof of the corresponding claim in the proof of Lemma 3.1. This completes the proof of Proposition 3.2. □

4. A technical lemma. Theorem 1 will be proved by constructing an infinite sequence of CE maps and then taking limits as in Proposition 3.2. The lemma in this section is the inductive step in that infinite construction. It is an ε-controlled version of [**St**] and corresponds to Proposition 6 of [**Hs**].

LEMMA 4.1. *Let X be a k-dimensional, LC^r compactum tamely embedded in $\mathbf{R}^m$, $m \geqslant 2k+3$, $r \leqslant k-2$, and let M be a PL n-manifold, $n \geqslant k+3$ and $n \geqslant 2k-r$. If ε_1, ε_2, α_1 and α_2 are positive numbers and*

$$X_1 \xleftarrow{\lambda_1} Z_1 \xrightarrow{\mu_1} Y_1$$

satisfies

(4.1) *X_1 is a k-dimensional polyhedron which is the spine of a strong (ε_1, r)-deformation neighborhood N_1 of X in $\mathbf{R}^m$;*

(4.2) *N_1 retracts to X_1 via an ε_1-retraction ρ_1: $N_1 \to X_1$;*
(4.3) *$Z_1 = X_1 \cup M_1$, where M_1 is $(r+2)$-dimensional;*
(4.4) *$M_1 \cap X_1$ is $(r+1)$-dimensional and collared in M_1;*
(4.5) *λ_1: $Z_1 \to X_1$ is a PL CE retraction with $\lambda_1(M_1) \subset M_1 \cap X_1$;*
(4.6) *Y_1 is a k-dimensional subpolyhedron of M;*
(4.7) *μ_1: $Z_1 \to Y_1$ is a PL CE map;*
(4.8) *if $A \subset X_1$ and* $\operatorname{diam}[A] < \varepsilon_1$, *then* $\operatorname{diam}[\mu_1(A)] < \alpha_1$;
(4.9) *if $A \subset N_1$ and* $\operatorname{diam}[A] < \varepsilon_2$, *then* $\operatorname{diam}[\mu_1\rho_1(A)] < \alpha_2$; *and*
(4.10) *P_1 is a regular neighborhood of Y_1 and q: $P_1 \to Y_1$ is a PL α_1-retraction;*
then there exists

$$\begin{array}{ccccc} X_2 & \overset{\lambda_2}{\leftarrow} & Z_2 & \overset{\mu_2}{\rightarrow} & Y_2 \\ \downarrow f & & \downarrow g & & \downarrow h \\ X_1 & \overset{\lambda_1}{\leftarrow} & Z_1 & \overset{\mu_1}{\rightarrow} & Y_1 \end{array}$$

which satisfies
(4.11) *X_2 is a k-dimensional polyhedron which is the spine of a strong (ε_2, r)-deformation neighborhood N_2 of X in N_1;*
(4.12) *$f = \rho_1|X_2$;*
(4.13) *$Z_2 = X_2 \cup M_2$, where M_2 is $(r+2)$-dimensional;*
(4.14) *$M_2 \cap X_2$ is $(r+1)$-dimensional and collared in M_2;*
(4.15) *λ_2: $Z_2 \to X_2$ is a PL CE retraction with $\lambda_2(M_2) \subset M_2 \cap X_2$;*
(4.16) *Y_2 is a k-dimensional polyhedron in P_1;*
(4.17) *$h = q|Y_2$;*
(4.18) *μ_2: $Z_2 \to Y_2$ is a PL CE map;*
(4.19) *$\mu_2|X_2$ is so close to $\mu_1\rho_1|X_2$ that if $A \subset X_2$ and* $\operatorname{diam}[A] < \varepsilon_2$, *then* $\operatorname{diam}[\mu_2(A)] < \alpha_2$;
(4.20) *$\lambda_1 g$ is $4(r+1)\varepsilon_1$-homotopic to $f\lambda_2$ in K_1; and*
(4.21) *$\mu_1 g$ is $8(r+1)\alpha_1$-homotopic to $h\mu_2$ in Y_1.*

PROOF. We may assume that $\varepsilon_2 \leqslant \varepsilon_1$. Let N_2 be a strong (ε_2, r)-deformation neighborhood of X in N_1 which ε_2-collapses to a k-dimensional polyhedron X_2. Let ρ_2: $N_2 \to X_2$ be the ε_2-retraction induced by the collapse of N_2 to X_2 and let $f = \rho_1|X_2$.

We next construct Z_2. It will be constructed in two steps. First we will lift the attaching map for M_1 to X_2 and form Z' by attaching M_1 to X_2. Then Z_2 will be formed by using an ε-controlled version of Stallings [**St**] to attach further mapping cylinders to Z'. The details follow.

Let β: $M_1 \cap X_1 \to X_1$ denote the inclusion map. Since N_1 is an (ε_1, r)-deformation neighborhood, there exists β': $M_1 \cap X_1 \to X_2$ such that β' is $2\varepsilon_1$-homotopic to β in N_1. Define $Z' = X_2 \cup_{\beta'} M_1$. Notice that $f\beta'$ is $4\varepsilon_1$-homotopic to β in X_1. Thus there is a map g: $Z' \to Z_1$ such that $g|X_2 = f$ and $d(x, g(x)) < 4\varepsilon_1$ for every $x \in M_1$. The map g is just f on X_2, the identity on M_1 minus a small collar on

$M_1 \cap X_1$ and stretches the collar out over the homotopy from β to $f\beta'$. Furthermore, λ_1 induces a retraction λ': $Z' \to X_2$ such that $\lambda_1 g$ and $f\lambda'$ are $4\varepsilon_1$-homotopic in X_1.

Consider $\mu_1 g$: $Z' \to Y_1 \subset P_1$. Let g': $Z' \to P_1$ be a PL general position map which is α_1-homotopic to $\mu_1 g$. Let Σ denote the singular set of g' and let M' denote the mapping cylinder of $g'|\Sigma$: $\Sigma \to g'(\Sigma)$. Then $\dim \Sigma \leqslant r$ and $\dim M' \leqslant r+1$. We identify Σ with $\Sigma \times \{0\} \subset M'$ and let ξ: $M' \to g'(\Sigma)$ denote the map which collapses out the fibers of the mapping cylinder.

We now wish to extend the inclusion map $\Sigma \subset Z'$ to a map of M' into Z'. The α_1-homotopy from g' to $\mu_1 g$ gives us a map τ_1: $M' \to P_1$ such that $\tau_1|\Sigma = \mu_1 g|\Sigma$, $\tau_1|g'(\Sigma) =$ inclusion and the diameter of the image under τ_1 of each fiber of M' is less than α_1. Then $q\tau_1$: $M' \to Y_1$ and $q\tau_1|\Sigma = \mu_1 g|\Sigma$. Notice that the α_1-homotopy from τ_1 to $q\tau_1$ can be used to construct an α_1-homotopy form $q\tau_1$ to ξ. The homotopy is obviously not rel Σ, but it does have the property that, when restricted to Σ, it is just the reverse of the α_1-homotopy from g' to $\mu_1 g$. Since μ_1 is a CE map, there exists a lift τ_2: $M' \to Z_1$ such that $\tau_2|\Sigma = g|\Sigma$ and such that $\mu_1\tau_2$ is α_1-homotopic to $q\tau_1$ rel Σ. We may assume that τ_2 is PL. The next step is to lift to Z'. We define τ_3: $M' \to Z'$ as follows. First $\tau_3|\Sigma =$ inclusion and $\tau_3|\tau_2^{-1}(M_1) = g^{-1}\tau_2|\tau_2^{-1}(M_1)$. Then use Proposition 2.4 to extend τ_3 to all of M' in such a way that $g\tau_3$ is $5\varepsilon_1$-homotopic to τ_2 rel Σ.

Now define $M'' = M' \times [0,1]/\{\{x\} \times [0,1] | x \in \Sigma\}$; i.e., M'' is obtained from $M' \times [0,1]$ by identifying each of the sets $\{x\} \times [0,1]$, $x \in \Sigma$, to a point. Notice that M'' is $(r+2)$-dimensional and contains two copies of M': $M' \times \{0\}$ and $M' \times \{1\}$. Roughly speaking, we will attach M'' to Z' along one copy of M' to form Z_2 and shrink out fibers in the other copy to form Y_2.

The map τ_3 defines, in a natural way, a map τ: $M' \times \{0\} \to Z'$. Define $Z'' = Z' \cup_\tau M'$, define ξ': $Z'' \to Z'$ to be the map which collapses out the interval factors in M'' and define g'': $Z'' \to Z_1$ by $g'' = g\xi'$. Then define Y' to be the polyhedron obtained from M'' by shrinking out fibers of the mapping cylinder $M' \times \{1\} \subset M''$. Let μ: $Z'' \to Y'$ be the natural collapsing map.

We now claim that there is a map π: $Y' \to P_1$ such that $q\pi\mu$ is $8\alpha_1$-homotopic to $\mu_1 g''$ in Y_1. Since the point preimages of g' are precisely the subsets of Z' which are identified by μ, we can use g' to define $\pi|Z'$. We can further extend π to $Z' \cup M' \times \{1\}$ by making $\pi(x,1) = \xi(x)$ for each $x \in M'$. In order to extend π to the rest of M'' we must find an $8\alpha_1$-homotopy of $g'\tau_3$ to ξ which is rel Σ.

First, there is an α_1-homotopy of $g'\tau_3$ to $\mu_1 g\tau_3$ (not rel Σ). Next, there is a $5\alpha_1$-homotopy (this one rel Σ) of $\mu_1 g\tau_3$ to $\mu_1\tau_2$. But then there is an α_1-homotopy (rel Σ) to $q\tau_1$. Finally, $q\tau_1$ is α_1-homotopic to ξ. All of this gives us an $8\alpha_1$-homotopy from $g'\tau_3$ to ξ. On Σ this homotopy is not constant, but is the homotopy from $g'|\Sigma$ to $\mu_1 g|\Sigma$ followed by the reverse of that same homotopy. We can use a small collar on Σ in M' to taper off that homotopy and arrive at the homotopy we seek which is rel Σ. Thus π: $Y' \to P_1$ is defined. We may assume that π is PL and in general position.

Notice that $q\pi\mu$: $Z'' \to Y_1$ and $\mu_1 g''$: $Z'' \to Y_1$ are $8\alpha_1$-homotopic. We can also define a CE map λ: $Z'' \to X_2$ by $\lambda = \lambda'\xi'$. Then the two maps $\lambda_1 g''$: $Z'' \to X_1$ and $f\lambda$: $Z'' \to X_1$ are $4\varepsilon_1$-homotopic in X_1.

Special case: $k < (2/3)(n-1)$. In this case the dimension of the singular set of π is negative and thus π is an embedding. We take $Z_2 = Z''$, $g = g''$, $Y_2 = \pi(Y')$, $\mu_2 = \pi\mu$, $\lambda_2 = \lambda$, and $h = q|Y_2$. Since $\mu_2|X_2 = g'|X_2$, we can easily make sure that conclusion (4.19) is satisfied and the proof is complete in this case.

General case. In general, the dimension of the singular set of π is less than or equal to $r-1$. We inductively apply the same construction to $\mu\pi$ as was applied above to g'. This gives new polyhedra Z''' and Y'' and a map π': $Y'' \to P_1$. This time we will only have to attach polyhedra of dimension $(r-1)+2 = r+1$, and so the dimension of the singular set of π' will be only $r-2$. This is repeated inductively until π is an embedding, which will take at most r repetitions. □

5. Proof of Theorem 1. In this section we use the lemmas from the preceding three sections to prove Theorem 1. The construction is based on that in the proof of Theorem 19 in **[Hs]**.

Let X, k, r, f, M, and n be as in the statement of Theorem 1. We begin by embedding X as a tame subset of $\mathbf{R}^{2k+3}$. The map f: $X \to M$ can be extended to a compact neighborhood U of X in $\mathbf{R}^{2k+3}$. We may assume that f: $U \to M$ is PL and in general position. Fix a sequence $\{\alpha_i\}$ of positive numbers such that $\sum \alpha_i < \infty$. Next choose a number $\varepsilon_1 > 0$ such that if A is a compact subset of U and $\operatorname{diam}[A] < \varepsilon_1$, then $\operatorname{diam}[f(A)] < \alpha_1$. Now let N_1 be a strong (ε_1, r)-deformation neighborhood of X in U. Let us say that N_1 ε_1-collapses to the k-dimensional polyhedron X_1 and that the collapse induces the ε_1-retraction ρ_1: $N_1 \to X_1$.

Since N_1 is a strong (ε_1, r)-deformation neighborhood of X, $f|X_1$: $X_1 \to M$ is $(r+1)$-connected. We can therefore apply Stallings' construction **[St]** to $f|X_1$. This gives us two k-dimensional polyhedra $Z_1 \supset X_1$ and $Y_1 \subset M$, a PL CE retraction λ_1: $Z_1 \to X_1$ and a PL CE map μ_1: $Z_1 \to Y_1$. Furthermore, Z_1 consists of X_1 with some mapping cylinders attached. In particular, we can write $Z_1 = X_1 \cup M_1$, where M_1 is $(r+2)$-dimensional, $M_1 \cap X_1$ is $(r+1)$-dimensional, and $M_1 \cap X_1$ is collared in M_1. We can also arrange that $\mu_1|X_1$ is arbitrarily close to $f|X_1$, and so we choose $\mu_1|X_1$ to be so close to $f|X_1$ that if $A \subset X_1$ and $\operatorname{diam}[A] < \varepsilon_1$, then $\operatorname{diam}[\mu_1(A)] < \alpha_1$.

We will next apply Lemma 4.1, but we must first specify the number ε_2 of the lemma. Choose $\varepsilon_2 > 0$ such that if $A \subset N_1$ and $\operatorname{diam}[A] < \varepsilon_2$, then $\operatorname{diam}[\mu_1\rho_1(A)] < \alpha_2$. Now apply Lemma 4.1. This gives us a new diagram

$$\begin{array}{ccccc} X_2 & \overset{\lambda_2}{\leftarrow} & Z_2 & \overset{\mu_2}{\rightarrow} & Y_2 \\ \downarrow f_1 & & \downarrow g_1 & & \downarrow h_1 \\ X_1 & \overset{\lambda_1}{\leftarrow} & Z_1 & \overset{\mu_1}{\rightarrow} & Y_1 \end{array}$$

as well as an ε_2-neighborhood N_2 of X_2.

Let $\varepsilon_3 > 0$ be a number such that if $A \subset N_2$ and $\operatorname{diam}[A] < \varepsilon_3$, then $\operatorname{diam}[\mu_2 \rho_2(A)] < \alpha_3$. Apply Lemma 4.1 again, this time to X_2, Z_2, and Y_2. This procedure is continued inductively and produces an infinite diagram:

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
X_4 & \stackrel{\lambda_4}{\leftarrow} & Z_4 & \stackrel{\mu_4}{\rightarrow} & Y_4 \\
\downarrow f_3 & & \downarrow g_3 & & \downarrow h_3 \\
X_3 & \stackrel{\lambda_3}{\leftarrow} & Z_2 & \stackrel{\mu_3}{\rightarrow} & Y_3 \\
\downarrow f_2 & & \downarrow g_2 & & \downarrow h_2 \\
X_2 & \stackrel{\lambda_2}{\leftarrow} & Z_2 & \stackrel{\mu_2}{\rightarrow} & Y_2 \\
\downarrow f_1 & & \downarrow g_1 & & \downarrow h_1 \\
X_1 & \stackrel{\lambda_1}{\leftarrow} & Z_1 & \stackrel{\mu_1}{\rightarrow} & Y_1
\end{array}
$$

We can make sure that the ε_i are chosen in such a way that $\sum \varepsilon_i < \infty$. Let $Z = \varprojlim \{Z_i, g_i\}$ and let $Y = \bigcap P_i$. By Proposition 3.2 there are CE maps $\lambda\colon Z \to X$ and $\mu\colon Z \to Y$. This completes the construction of Y. To finish the proof of the theorem we need only show that Y is LC^r. To do so, we prove the following statement:

Suppose L is an $(r+1)$-dimensional polyhedron and L_0 is a subpolyhedron of L. For every map $F_0\colon L_0 \to Y_{i+1}$ such that $h_i F_0$ extends to $F\colon L \to Y_i$ there exists an extension $F'\colon L \to Y_{i+1}$ of F_0 such that $h_i F'$ is $[5 + 12(r+1)]\alpha_i$-homotopic to F rel L_0 in Y_i.

This is really just the statement that Proposition 2.4 holds for the neighborhoods $\{P_i\}$ of Y. It is easy to see that this implies that Y is LC^r: just take a small singular m-sphere in Y, $m \leqslant r$, homotope it to a point in a small subset of M, and then apply the statement above to push that homotopy into Y by pushing it into smaller and smaller neighborhoods of Y with a Cauchy sequence of maps.

We will indicate how to get the extension F' whose existence was claimed above. It is constructed by using the CE maps λ_i and μ_i to make the transition from Y to X and back again. We leave the calculation of the size of the homotopies involved to the reader.

Since μ_{i+1} is CE, we can lift F_0 to a map $F_1\colon L_0 \to Z_{i+1}$ such that $\mu_{i+1} F_i$ is arbitrarily close to F_0. We choose it to be so close that if we can extend the map $\mu_{i+1} F_i$, then we are done. Consider $\mu_i g_i F_1\colon L_0 \to Y_i$. This map is $8(r+1)\alpha_i$-homotopic to $h_i \mu_{i+1} F_i$ which extends to all of L (via F) and thus there is an extension $F_2\colon L \to Z_i$ of $g_i F_1$. Now consider $\lambda_{i+1} F_1\colon L_0 \to X_{i+1}$. Since $f_i \lambda_{i+1} F_1$ is $4\varepsilon_1$-homotopic to $\lambda_i g_i F_1$ and $\lambda_i g_i F_1$ extends to $\lambda_i F_2$, we have that $f_i \lambda_{i+1} F_1$ extends to $F_3\colon L \to X_i$. Thus we are in a position to apply Proposition 2.4 to the maps $\lambda_{i+1} F_1\colon L_0 \to X_{i+1}$ and $F_3\colon L \to X_i$ which extends $f_i \lambda_{i+1} F_1$. Thus there is an extension F_4: $L \to X_{i+1}$ of $\lambda_{i+1}F_1$ to all of L. Now use the fact that λ_{i+1} is CE to lift F_4 to a map F_5: $L \to Z_{i+1}$ which extends F_1. Now $\mu_{i+1}F_5$ is the desired extension of $\mu_{i+1}F_1$. □

REMARK 5.1. Notice that (in the proof above) $\varprojlim\{X_i, f_i\} \subset \varprojlim\{Z_i, g_i\}$. So if we did the extra work necessary to make the map G of Lemma 3.1 a homeomorphism, we would have $X \subset Z$ and λ: $Z \to X$ would be a retraction.

REMARK 5.2. It might seem more natural to assume in the statement of Theorem 1 that f is $(r+1)$-shape connected rather than $(r+1)$-connected. But the two assumptions are equivalent because either way we get that the extended map f has the property that $f|X_i$: $X_i \to M$ is $(r+1)$-connected (cf. [**Hs**, Proposition 14]).

REFERENCES

[**Br**] M. Brown, *Some applications of an approximation theorem for inverse limits*, Proc. Amer. Math. Soc. **11** (1960), 478–483.

[**Ch**] T. A. Chapman, *Cell-like mappings*, Algebraic and Geometrical Methods in Topology, Lecture Notes in Math., vol. 428, Springer-Verlag, New York, 1974, pp. 230–240.

[**C-I**] D. Coram and I. Ivanšić, *Shape embedding theorem in* (*co*)*dimension two*, preprint.

[**D-H**] P. Duvall and L. S. Husch, *A continuum of dimension n which does not embed up to shape in* 2*n-space*, Proc. Internat. Conf. on Geometric Topology, Polish Scientific Publishers, Warsaw, 1980, pp. 113–119.

[**D-S**] J. Dydak and J. Segal, *Shape theory*: *An introduction*, Lecture Notes in Math., vol. 688, Springer-Verlag, New York, 1978.

[**Ed**] R. D. Edwards, *Demension theory*. I, Geometic Topology, Lecture Notes in Math., vol. 438, Springer-Verlag, New York, 1975, pp. 195–211.

[**Fe1**] S. Ferry, *A stable converse to the Vietoris-Smale theorem with applications to shape theory*, Trans. Amer. Math. Soc. **261** (1980), 369–386.

[**Fe2**] ______, *Homotopy, simple homotopy and compacta*, Topology **19** (1980), 101–110.

[**Hd**] J. F. P. Hudson, *Piecewise linear embeddings*, Ann. of Math. (2) **85** (1967), 1–31.

[**Hs**] L. S. Husch, *UV^k maps and embeddings*, Topology Appl. **15** (1983), 273–291.

[**H-I**] L. S. Husch and I. Ivanšić, *Embedding compacta up to shape*, Shape Theory and Geometric Topology, Lecture Notes in Math., vol. 870, Springer-Verlag, New York, 1981, pp. 119–134.

[**Ir**] M. C. Irwin, *Embeddings of polyhedral manifolds*, Ann. of Math. (2) **82** (1965), 1–14.

[**La**] R. C. Lacher, *Cell-like mappings*. I, Pacific J. Math. **30** (1969), 717–731.

[**L-V**] V. T. Liem and G. A. Venema, *Stable shape concordance implies homeomorphic complements*, Fund. Math. (to appear).

[**St**] J. Stallings, *The embedding of homotopy types into manifolds*, mimeographed notes, Princeton University, 1965.

[**Št**] M. A. Štańko, *The embedding of compacta in Euclidean space*, Mat. Sb. **83 (125)** (1970), 234–255.

[**Wh**] H. Whitney, *The self-intersections of a smooth n-manifold in* 2*n-space*, Ann. of Math. (2) **43** (1944), 220–246.

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